DIVERGENCE-PRESERVING GEODESIC SYMMETRIES

J. E. D'ATRI & H. K. NICKERSON

On a differentiable manifold of dimension n, a local volume element ω (non-vanishing n-form) determines the divergence, relative to ω, of a local vector field Y by

$$(1) \qquad (\text{div } Y)\omega = L_Y\omega .$$

On a Riemannian manifold, there are two canonical choices of ω on any simply-connected neighborhood, differing only in sign, corresponding to the two possible orientations: ω is defined by the condition that any positively-oriented orthonormal frame shall span unit volume. By (1), either of these choices determines the same local map $Y \to \text{div } Y$.

In this paper, we consider Riemannian spaces in which every local geodesic symmetry is divergence-preserving (or, equivalently, volume-preserving up to sign). This class of spaces obviously includes the Riemannian locally symmetric spaces, in which every local geodesic symmetry satisfies the stronger condition of being an isometry. It also includes the harmonic spaces with positive-definite metric. An example is given, which shows that our class is strictly larger than either of these subclasses.

We derive an infinite sequence of necessary conditions on the curvature (sufficient in the case of an analytic manifold), which are a subset of the necessary conditions for a harmonic space.

1. Equivalent statements of the property

All Riemannian structures, functions, vector fields, mappings, etc., will be assumed differentiable of class C^∞. In so far as possible, the notation will follow [2]. The notations $\Lambda, \Omega,$ and Π are chosen to agree with [4].

Lemma 1.1. *A local diffeomorphism is divergence-preserving if and only if it preserves volume to within a constant factor.*

Proof. Let $F: M \to N$ be a local diffeomorphism, and define the non-zero local scalar function h on M by $F^*\tilde{\omega} = h\omega$, where $\omega, \tilde{\omega}$ are given local volume elements on M, N respectively. If Y is a local vector field on M, then

$$(\text{div } F_*Y) \circ F = \text{div } Y + (Y \cdot h)/h .$$

In order that $(\text{div } F_* Y) \circ F = \text{div } Y$ for all Y, it is clearly necessary and sufficient that $Y \cdot h$ vanish for all Y, or that the function h be constant.

We shall apply the above lemma in the case that $N = M$, where M is a Riemannian manifold, with F the local geodesic symmetry about a point $m \in M$, corresponding to the Riemannian connection, and with $\tilde{\omega} = \omega$, where ω is either of the two canonical local volume elements near m. Since the geodesic symmetry maps each tangent vector at m into its negative, the constant factor h in Lemma 1.1 can only be $(-1)^n$, where n is the dimension of M.

We restrict to a normal neighborhood U of m such that each point p in U is joined to m by a unique geodesic lying in U and such that the geodesic symmetry with pole m is a diffeomorphism (involutive) of U onto itself. We restrict the parameter on geodesics through m to s, where s is the arc length along a directed geodesic, with $s = 0$ at m. Then the geodesic symmetry, with pole m, can be described as $s \to -s$. That is, a point p corresponding to the parameter value s on a directed geodesic through m is mapped to the point, on the same geodesic, corresponding to the parameter value $-s$. In particular, a function defined on U is preserved by the geodesic symmetry if and only if its restriction to each geodesic through m is an *even* function of the parameter s.

A normal coordinate system $(y^1, \cdots, y^n)$ can be introduced in U, in which the parametric equations of a geodesic issuing from m are given by $y^i = a^i s$, $i = 1, \cdots, n$, $s \geq 0$, where the constants a^i are the components of the initial vector at m relative to a positively-oriented orthonormal basis for the tangent vectors at m. Then the local volume element ω on U is expressed by the classical formula

$$(2) \qquad\qquad \omega = \sqrt{g} \, dy^1 \wedge \cdots \wedge dy^n \, ,$$

where g is the determinant of the symmetric matrix expressing the Riemannian metric in these local coordinates, with $g(m) = 1$. The function g is independent of the positively-oriented normal coordinate system chosen since a change of coordinates within this class is effected by a constant orthogonal matrix with determinant 1.

In this section and through equation (8) of the next section, we reserve the notation X for the unit tangent vector, at $p \in U$, along the directed geodesic from m to p. The unit vectors X give a vector field (with singularity at m). The vector field sX, $s \geq 0$, is non-singular and is preserved by the geodesic symmetry about m. In any normal coordinate system, $sX = y^i \partial / \partial y^i = s \, d/ds$ (summation convention assumed) so

$$(3) \qquad\qquad \text{div } sX = n + s \frac{d}{ds} (\log \sqrt{g}) \, .$$

For the scalar function Ω defined by $\Omega(p) = s^2/2$, where s is the distance from m to p, we have $sX = \text{grad } \Omega$, and

(4) $\mathrm{div}\, sX = \mathrm{div}\,\mathrm{grad}\,\Omega = \Delta\Omega$,

where Δ is the Laplacian (with classical choice of sign). Finally, if Λ is the field of linear transformations on the tangent vectors given by

$$\Lambda(Y) = \nabla_Y(sX) ,$$

then

(5) $\mathrm{div}\, sX = \mathrm{trace}\,\Lambda$,

([2, I, p. 282], since $\Lambda = -A_{sX} = -L_{sX} + \nabla_{sX}$). For later reference, we note that

(6) $\Lambda = I$ at m.

Theorem 1.2. *Let m be a point of the Riemannian manifold M. Then the following are equivalent*:
  (a)   *The local geodesic symmetry about m is divergence-preserving.*
  (b)   *The function g in* (2) *is even.*
  (c)   *The function* $\mathrm{div}\, sX$ *is even.*
  (d)   *The function* $\Delta\Omega$ *is even.*
  (e)   *The function* $\mathrm{trace}\,\Lambda$ *is even.*

Proof.   The equivalence of (b), (c), (d), and (e) follows from the formulas (3), (4), and (5) for $\mathrm{div}\, sX$. The equivalence of (a) and (b) follows from Lemma 1.1 and the fact that the geodesic symmetry sends $dy^1 \wedge \cdots \wedge dy^n$ in (2) into $(-1)^n dy^1 \wedge \cdots \wedge dy^n$.

Thus, the class of Riemannian manifolds M considered in this paper are those for which any one of the equivalent properties above can be verified for each $m \in M$. The necessary conditions on the curvature (§ 2) will be drived from (e). The example of § 3 will be checked by (b). The subclass of harmonic spaces is defined by the stronger requirement that the even functions in the statements (b), (c), (d), or (e) above be the same function of s on all geodesics through m.

2.   Conditions on the curvature

For any vector field Y, we have

(7)
$$R(Y, sX)sX = \nabla_Y\nabla_{sX}(sX) - \nabla_{sX}\nabla_Y(sX) - \nabla_{[Y, sX]}(sX)$$
$$= -(\nabla_{sX}\Lambda)(Y) + \Lambda(Y) - \Lambda(\Lambda(Y)) .$$

For $s < 0$, the field sX is the negative of that used in deriving (7). In this case, we have

(7′)
$$R(Y, sX)sX = R(Y, -sX)(-sX)$$
$$= -(\nabla_{-sX}\Lambda)(Y) + \Lambda(Y) - \Lambda(\Lambda(Y)) .$$

The linear transformations Π are defined by

$$(8) \qquad s^2\Pi(Y) = -R(Y, sX)sX = -R(Y, -sX)(-sX) \ .$$

If we change notation (for $s < 0$) so that X is the unit tangent along a directed geodesic through m, then (7) yields the differential equation [3], [4]

$$(9) \qquad s\nabla_X\Lambda = \Lambda - \Lambda \circ \Lambda + s^2\Pi$$

for Λ along a directed geodesic through m. From (6), the initial condition is

$$(10) \qquad \Lambda|_0 = I \qquad \text{at } s = 0 \ .$$

The equation for trace Λ is

$$s\frac{d}{ds} \text{trace } \Lambda = \text{trace } s\nabla_X\Lambda = \text{trace } \Lambda - \text{trace } \Lambda \circ \Lambda + s^2 \text{ trace } \Pi \ ,$$

where $-\text{trace }\Pi$ is the same as the value of $S(X, X)$ for the Ricci tensor S.

Along a fixed geodesic through m, the non-linear first order differential equation (9), with initial condition (10) at $s = 0$, determines the transformations completely, including the values of the derivatives $\nabla_X^r\Lambda|_0$ for $r = 1, 2, \cdots$. We set $\nabla_X^0\Lambda = \Lambda$.

Operating by ∇_X on both sides of (9), one obtains

$$s\nabla_X^2\Lambda + \nabla_X\Lambda = \nabla_X\Lambda - \nabla_X\Lambda \circ \Lambda - \Lambda \circ \nabla_X\Lambda + s^2\nabla_X\Pi + 2s\Pi \ ,$$

which, with (10), implies

$$(11_1) \qquad \nabla_X\Lambda|_0 = 0 \ .$$

With further differentiation and the use of Leibniz formulas and (11_1), one obtains the recurrence formula

$$(12_r) \qquad (r + 1)\nabla_X^r\Lambda|_0 = r(r - 1)\nabla_X^{r-2}\Pi|_0 - \sum_{q=2}^{r-2}\binom{r}{q}\nabla_X^q\Lambda \circ \nabla_X^{r-q}\Lambda|_0$$

of Ledger [3], [4]. These give

$$(11_r) \qquad \nabla_X^r\Lambda|_0 = \sum c_{i_1\cdots i_k}^r\nabla_X^{i_1}\Pi \circ \cdots \circ \nabla_X^{i_k}\Pi|_0 \ ,$$

where the (absolute) constants $c_{i_1\cdots i_k}^r$, defined only for $r = i_1 + \cdots + i_k + 2k \geq 2$, $i_j \geq 0$, $1 \leq k \leq [r/2]$, are given by

$$(13_1) \qquad c_{i_1}^r = c_{r-2}^r = \frac{r(r - 1)}{r + 1} \ ,$$

and, for $k \geq 2$,

$$(13_k) \qquad c^r_{i_1 \cdots i_k} = -\frac{1}{r+1} \sum_{j=1}^{k} \binom{r}{q_j} c^{q_j}_{i_1 \cdots i_j} c^{r-q_j}_{i_{j+1} \cdots i_k}$$

with

$$q_j = 2j + (i_1 + \cdots + i_j) = r - 2k + 2j - (i_{j+1} + \cdots + i_k) \ .$$

If trace Λ is to be an even function of s along the geodesic, then the derivatives

$$\frac{d^r}{ds^r} \text{ trace } \Lambda = \text{trace } \nabla_X^r \Lambda$$

must vanish at $s = 0$ for odd values of r.

We define

$$(14_r) \qquad P^r = \sum c^r_{i_1 \cdots i_k} \nabla_X^{i_1} \Pi \circ \cdots \circ \nabla_X^{i_k} \Pi \ .$$

Then, by (11_r), the conditions are trace $P^r|_m = 0$ for odd values of r. However, the transformations P^r are defined at all points of the geodesic through m and depend only on the curvature and the choice of geodesic, by (8), so the conditions hold for all points m on the geodesic. Hence

Theorem 2.1. *If every local geodesic symmetry on a Riemannian manifold is divergence-preserving, then the "curvature" transformations P^r associated with an arbitrary geodesic satisfy*

$$(15) \qquad \text{trace } P^r = 0, \qquad r = 3, 5, 7, \cdots .$$

Theorem 2.2. *If the manifold is real analytic, the above necessary conditions are sufficient.*

Proof. In this case, trace Λ has a series expansion in powers of s along any geodesic through m. The conditions (15) in the formulas (11_r) ensure that only even powers will occur, so trace Λ is an even function of s.

The necessary conditions of Theorem 2.1 can be expressed invariantly as the vanishing of certain tensors on the manifold. One approach (cf. [4]) is to note that $(r-1)$trace P^r is the same as $A^r(X, \cdots, X)$ where A^r is the contraction of the $(r+1)$st normal tensor of Veblen and [4, p. 13] the contracted tensor is symmetric in the r entries occupied by the unit tangent vector X to the geodesic. Then the invariant condition is that the contracted normal tensor vanish for $r = 3, 5, 7, \cdots$.

Another way is to work out the P^r in terms of the curvature R and its covariant derivatives, using the relation

$$(16) \qquad (\nabla_X^r \Pi)(Y) = (\nabla_X^r R)(Y, X)X = (\nabla^r R)(Y, X; X, \cdots, X)X \ ,$$

which can be proved inductively, using the fact that $\nabla_X X = 0$ when X is the

unit tangent vector along a geodesic. If (16) is substituted into (14_r), the conditions (15) become conditions expressed in terms of R and its covariant derivatives. The invariant form of the conditions is then obtained by polarization.

For example, the condition trace $P^3 = 0$ gives trace $\nabla_X \Pi = 0$ or $(\nabla_X S)(X, X) = 0$. In polarized form, this becomes

$$(17) \qquad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0$$

for arbitrary X, Y, Z. This condition holds also for harmonic spaces, but is there a consequence of the stronger condition that trace $P^2|_m$ is independent of the geodesic through m. The following two results require only the weaker condition (17).

Proposition 2.3. *A Riemannian manifold satisfying* (17) *has constant scalar curvature.*

Proof. Computed in an orthonormal frame $\{X_1, \cdots, X_n\}$, the scalar curvature K is given by

$$K = \sum_{j=1}^n S(X_j, X_j) \ .$$

The Bianchi identities give

$$X_i \cdot K = \sum_{j=1}^n (\nabla_{X_i} S)(X_j, X_j) = 2 \sum_{j=1}^n (\nabla_{X_j} S)(X_j, X_i) \ .$$

However, the condition (17) implies that

$$(\nabla_{X_i} S)(X_j, X_j) = -2(\nabla_{X_j} S)(X_j, X_i) \ ,$$

so $X_i \cdot K$ must vanish for $i = 1, \cdots, n$.

Corollary 2.4. *A Riemannian manifold of dimension* 2 *satisfying* (17) *has constant curvature.*

The additional condition implied by trace $P^5 = 0$, assuming (17), is that

$$\text{trace } \nabla_X (\Pi \circ \Pi) = 0 \ ,$$

and then trace $P^7 = 0$ requires

$$\text{trace } \nabla_X (32\Pi \circ \Pi \circ \Pi + 9\nabla_X \Pi \circ \nabla_X \Pi) = 0 \ .$$

The polarized forms of these conditions are omitted since we have no geometric interpretations of the conditions.

We have verified that naturally reductive homogeneous space with positive-definite metric satisfy the necessary conditions for $r = 3, 5, 7$. The computation required the use of the property $c_{i_1 \cdots i_k}^r = c_{i_k \cdots i_1}^r$. The example given in § 3 is a special case of this class.

3. Example

For a Riemannian homogeneous space M, it suffices to verify any one of the equivalent properties listed in Theorem 1.2 at a single point $m \in M$ in order to know that the property holds for all points of M.

A one-parameter family of examples, of dimension 3, was constructed by consulting [1]. The parameterization is by real numbers α and β satisfying $\alpha^2 + \beta^2 = 1$. For $\alpha \neq 0$, $\beta \neq 0$, the spaces are diffeomorphic to S^3 but not symmetric [1] and therefore not harmonic either. For $\alpha = 0$, the space is isometric to the standard S^3 and therefore both symmetric and harmonic. For $\beta = 0$, the space is isometric to $S^2 \times \mathbf{R}$.

We take $M = G/H$ where $G = SU(2) \times \mathbf{R}^+$ with elements consisting of pairs

(18) $$\begin{pmatrix} \eta & \zeta \\ -\bar{\zeta} & \bar{\eta} \end{pmatrix}, e^r ; \qquad \eta\bar{\eta} + \zeta\bar{\zeta} = 1, r \in \mathbf{R} ,$$

and H is the subgroup of pairs of the form

$$\begin{pmatrix} e^{-i\alpha h} & 0 \\ 0 & e^{i\alpha h} \end{pmatrix}, e^{\beta h} ; \qquad h \in \mathbf{R} .$$

Then $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathbf{R}$. The $\mathrm{ad}(G)$-invariant positive-definite symmetric bilinear form $B: \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ is taken to be $-1/8$ the Killing form on $\mathfrak{su}(2)$ and the euclidean scalar product on $\mathbf{R}$. An orthonormal basis for $\mathfrak{g}$ with respect to this form is given by

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + 0 , \qquad Y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + 0 ,$$

$$Z = \beta\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \alpha , \qquad W = -\alpha\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \beta .$$

Here W is a basis for the Lie algebra $\mathfrak{h}$ of H. We set $\mathfrak{m} = $ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$, spanned by X, Y, Z, and take the G-invariant metric on $M = G/H$ to be that induced by restricting B to $\mathfrak{m}$. Then [2, II, p. 203] M is naturally reductive with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. In particular, [2, II, p. 197, p. 192], if $\pi: G \to G/H$ sends g into gH, then the geodesics through $m = \pi(e)$ are given by $\pi(\exp sU)$ for $U \in \mathfrak{m}$.

Next we note that $\exp: \mathfrak{g} \to G$ sends $xX + yY + zZ + hW$ into an element of the form (18) with

$$\operatorname{Re}\eta = \cos\delta , \qquad \delta^2 = x^2 + y^2 + (\beta z - \alpha h)^2 ,$$

(19) $$\operatorname{Im}\eta = (\beta z - \alpha h)\frac{\sin\delta}{\delta} ,$$

$$\zeta = (x + iy)\frac{\sin\delta}{\delta} , \qquad r = \alpha z + \beta h .$$

For sufficiently small x, y, z, the projection π gives a diffeomorphism of $N = \{\exp(xX + yY + zZ)\}$ onto a neighborhood of $\pi(e)$ in M. Moreover, if $\pi(\exp(xX + yY + zZ))$ is assigned the coordinates (x, y, z), these coordinates are normal coordinates in the neighborhood $\pi(N)$.

Next we compute the orthonormal frame $\bar{X}, \bar{Y}, \tilde{Z}$ induced from X, Y, Z by the G-invariant structure (e.g., $\bar{X}(gH) = \pi_* L_g X$) in terms of the normal coordinates on $\pi(N)$. If $p = \pi a$, where $a = \exp(xX + yY + zZ) \in N$, then

$$\bar{X}(p) = \pi_* L_a X = \pi_* \frac{d}{dt} a \exp tX|_{t=0} = \frac{d}{dt} \pi(a \exp tX)|_{t=0}$$

$$= \frac{d}{dt} \pi(a(\exp tX)(\exp h(t)W))|_{t=0} ,$$

where $h(t)$ is uniquely and differentiably determined, for small t, by the conditions that $h(0) = 0$ and that $a(\exp tX)(\exp h(t)W)$ lies in N, that is,

$$(20) \qquad a(\exp tX)(\exp h(t)W) = \exp(x(t)X + y(t)Y + z(t)Z)$$

for suitable $x(t), y(t), z(t)$ such that $x(0) = x, y(0) = y, z(0) = 0$. Then

$$\bar{X}(p) = x'(0)\frac{\partial}{\partial x} + y'(0)\frac{\partial}{\partial y} + z'(0)\frac{\partial}{\partial z} .$$

Explicit equations relating $x(t), y(t), z(t)$ and $h(t)$ are obtained by computing the condition (20), using (19). Although these equations are not readily solvable, they determine the values of the derivatives at $t = 0$ explicitly. We set

$$A(x, y, z) = \frac{\delta \cos \delta - \sin \delta}{\delta^2 \sin \delta} , \qquad \delta^2 = x^2 + y^2 + \beta^2 z^2 .$$

If $\beta \neq 0$, then

$$\bar{X}(x, y, z) = a_1 \frac{\partial}{\partial x} + b_1 \frac{\partial}{\partial y} + c_1 \frac{\partial}{\partial z}$$

with

$$c_1 = -\beta \frac{A\beta xz + y}{1 + A\alpha^2(x^2 + y^2)} ,$$

$$a_1 = \frac{\alpha^2}{\beta}(A\beta xz - y)c_1 + 1 + A(y^2 + \beta^2 z^2) ,$$

$$b_1 = \frac{\alpha^2}{\beta}(A\beta yz + x)c_1 + \beta z - Axy .$$

Similarly, we obtain

$$\bar{Y}(x, y, z) = a_2\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y} + c_2\frac{\partial}{\partial z}$$

with

$$c_2 = -\beta\frac{A\beta yz - x}{1 + A\alpha^2(x^2 + y^2)} \, ,$$

$$a_2 = \frac{\alpha^2}{\beta}(A\beta xz - y)c_2 - \beta z - Axy \, ,$$

$$b_2 = \frac{\alpha^2}{\beta}(A\beta yz + x)c_2 + 1 + A(x^2 + \beta^2 z^2) \, ,$$

and

$$\tilde{Z}(x, y, z) = a_3\frac{\partial}{\partial x} + b_3\frac{\partial}{\partial y} + c_3\frac{\partial}{\partial z}$$

with

$$c_3 = \frac{1 + A(x^2 + y^2)}{1 + A\alpha^2(x^2 + y^2)} \, ,$$

$$b_3 = \frac{\alpha^2}{\beta}(A\beta xz - y)c_3 - \frac{1}{\beta}(A\beta xz - y) \, ,$$

$$a_3 = \frac{\alpha^2}{\beta}(A\beta yz + x)c_3 - \frac{1}{\beta}(A\beta yz + x) \, .$$

The orthonormal frame $\bar{X}, \bar{Y}, \tilde{Z}$ spans volume ± 1, depending on the orientation chosen. Consequently, the function $\sqrt{g}$ in (2), corresponding to these normal coordinates, is $\pm$ the reciprocal of

(21)
$$\det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \, ,$$

so it is sufficient to verify that the determinant in (21) is an even function of the normal coordinates (x, y, z). After subtraction of suitable multiples of the third column from the first and second columns, the evaluation of (21) is reduced to evaluating

(22)
$$-\frac{1}{\beta}\det\begin{pmatrix} 1 + A(y^2 + \beta^2 z^2) & \beta z - Axy & c_1 \\ -\beta z - Axy & 1 + A(x^2 + \beta^2 z^2) & c_2 \\ A\beta xz - y & A\beta yz + x & -\beta c_3 \end{pmatrix} \, .$$

Expanding (22) by the third column, we note that both c_3 and its minor are even, and then check that the odd terms in c_1(minor of c_1) $- c_2$(minor of c_2) cancel out.

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